27. $z = \dfrac{y^2}{4} - \dfrac{x^2}{9}$

28. $y^2 = x^2 + z^2$

29. $4x^2 - 3y^2 + 2z^2 = 0$

30. $\dfrac{x^2}{9} + \dfrac{y^2}{12} + \dfrac{z^2}{9} = 1$

31. $x^2 + y^2 + z^2 + 4x - by + 9z - b = 0$, where b is a constant

32. Using polar coordinates, describe the level curves of the function defined by

$$f(x, y) = 2xy/(x^2 + y^2) \text{ if } (x, y) \neq (0, 0) \text{ and } f(0, 0) = 0.$$

33. Let $f\colon \mathbb{R}^2\backslash\{\mathbf{0}\} \to \mathbb{R}$ be given in polar coordinates by $f(r, \theta) = (\cos 2\theta)/r^2$. Sketch a few level curves in the xy plane. Here, $\mathbb{R}^2\backslash\{\mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} \neq \mathbf{0}\}$.

34. Show that in Figure 2.1.15, the level "curve" $z = 3$ consists of two points.

2.2 Limits and Continuity

This section develops the concepts of open sets, limits, and continuity; open sets are needed to understand limits, and limits are in turn needed to understand continuity and differentiability.

As in elementary calculus, it is not necessary to completely master the limit concept in order to work problems in differentiation. For this reason, instructors may treat the following material with varying degrees of rigor. The student should consult with the instructor about the depth of understanding required.

Open Sets

We begin formulating the concept of an open set by defining an open disk. Let $\mathbf{x}_0 \in \mathbb{R}^n$ and let r be a positive real number. The ***open disk*** (or ***open ball***) of radius r and center $\mathbf{x}_0$ is defined to be the set of all points $\mathbf{x}$ such that $\|\mathbf{x} - \mathbf{x}_0\| < r$. This set is denoted $D_r(\mathbf{x}_0)$, and is the set of points $\mathbf{x}$ in $\mathbb{R}^n$ whose distance from $\mathbf{x}_0$ is less than r. Notice that we include only those $\mathbf{x}$ for which *strict* inequality holds. The disk $D_r(\mathbf{x}_0)$ is illustrated in Figure 2.2.1 for $n = 1, 2, 3$. For the case $n = 1$ and $x_0 \in \mathbb{R}$, the open disk $D_r(x_0)$ is the open interval $(x_0 - r, x_0 + r)$, which consists of all numbers $x \in \mathbb{R}$ *strictly* between $x_0 - r$ and $x_0 + r$. For the case $n = 2$, $\mathbf{x}_0 \in \mathbb{R}^2$, $D_r(\mathbf{x}_0)$ is the "inside" of the disk of radius r centered at $\mathbf{x}_0$. For the case $n = 3$, $\mathbf{x}_0 \in \mathbb{R}^3$, $D_r(\mathbf{x}_0)$ is the part strictly "inside" of the ball of radius r centered at $\mathbf{x}_0$.

DEFINITION: Open Sets Let $U \subset \mathbb{R}^n$ (that is, let U be a subset of $\mathbb{R}^n$). We call U an ***open set*** when for every point $\mathbf{x}_0$ in U there exists some $r > 0$ such that $D_r(\mathbf{x}_0)$ is contained within U; symbolically, we write $D_r(\mathbf{x}_0) \subset U$ (see Figure 2.2.2).

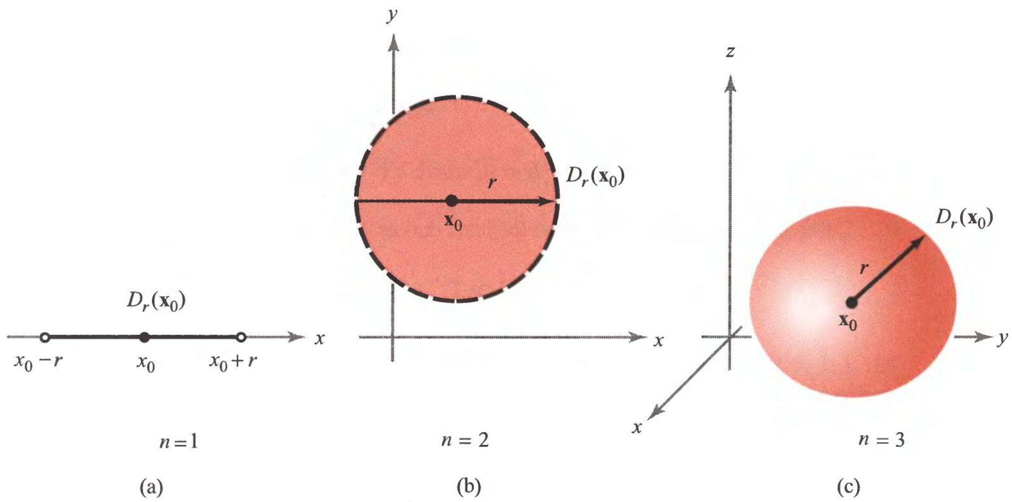


Figure 2.2.1 What disks $D_r(\mathbf{x}_0)$ look like in (a) one, (b) two, and (c) three dimensions.

The number $r > 0$ can depend on the point $\mathbf{x}_0$, and generally r will shrink as $\mathbf{x}_0$ gets closer to the "edge" of U. Intuitively speaking, a set U is open when the "boundary" points of U do not lie in U. In Figure 2.2.2, the dashed line is *not* included in U.

We establish the convention that the empty set $\emptyset$ (the set consisting of no elements) is open.

We have defined an open disk and an open set. From our choice of terms it would seem that an open disk should also be an open set. A little thought shows that this fact requires some proof. The following theorem does this.

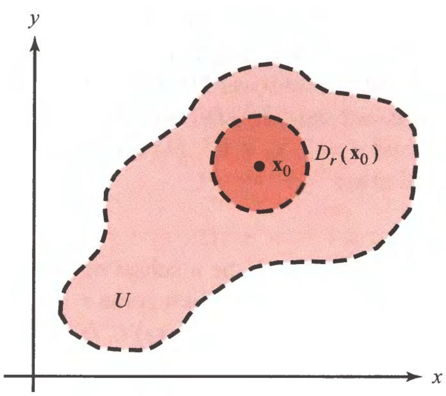


Figure 2.2.2 An open set U is one that completely encloses some disk $D_r(\mathbf{x}_0)$ about each of its points $\mathbf{x}_0$.

THEOREM 1 For each $\mathbf{x}_0 \in \mathbb{R}^n$ and $r > 0$, $D_r(\mathbf{x}_0)$ is an open set.

PROOF Let $\mathbf{x} \in D_r(\mathbf{x}_0)$, that is, let $\|\mathbf{x} - \mathbf{x}_0\| < r$. According to the definition of an open set, we must find an $s > 0$ such that $D_s(\mathbf{x}) \subset D_r(\mathbf{x}_0)$. Referring to Figure 2.2.3, we see that $s = r - \|\mathbf{x} - \mathbf{x}_0\|$ is a reasonable choice; note that $s > 0$, but that s becomes smaller if $\mathbf{x}$ is nearer the edge of $D_r(\mathbf{x}_0)$.

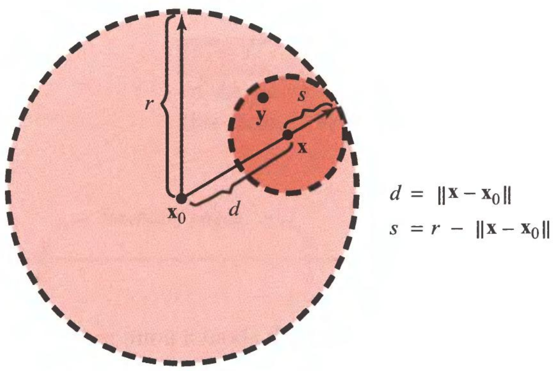


Figure 2.2.3 The geometry of the proof that an open disk is an open set.

To prove that $D_s(\mathbf{x}) \subset D_r(\mathbf{x}_0)$, let $\mathbf{y} \in D_s(\mathbf{x})$; that is, let $\|\mathbf{y} - \mathbf{x}\| < s$. We want to prove that $\mathbf{y} \in D_r(\mathbf{x}_0)$ as well. Proving this, in view of the definition of an r-disk, entails showing that $\|\mathbf{y} - \mathbf{x}_0\| < r$. This is done by using the triangle inequality for vectors in $\mathbb{R}^n$:

$$\|\mathbf{y} - \mathbf{x}_0\| = \|(\mathbf{y} - \mathbf{x}) + (\mathbf{x} - \mathbf{x}_0)\| \leq \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{x}_0\| < s + \|\mathbf{x} - \mathbf{x}_0\| = r.$$

Hence, $\|\mathbf{y} - \mathbf{x}_0\| < r$. ■

The following example illustrates some techniques that are useful in establishing the openness of sets.

EXAMPLE 1 Prove that $A = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$ is an open set.

SOLUTION The set is pictured in Figure 2.2.4.

Intuitively, this set is open, because no points on the "boundary," $x = 0$, are contained in the set. Such an argument will often suffice after one becomes accustomed to the concept of openness. At first, however, we should give details. To prove that A is open, we show that for every point $(x, y) \in A$ there exists an $r > 0$ such that $D_r(x, y) \subset A$. If $(x, y) \in A$, then $x > 0$. Choose $r = x$. If $(x_1, y_1) \in D_r(x, y)$, we have

$$|x_1 - x| = \sqrt{(x_1 - x)^2} \leq \sqrt{(x_1 - x)^2 + (y_1 - y)^2} < r = x,$$

and so $x_1 - x < x$ and $x - x_1 < x$. The latter inequality implies $x_1 > 0$, that is, $(x_1, y_1) \in A$. Hence $D_r(x, y) \subset A$, and therefore A is open (see Figure 2.2.5). ▲

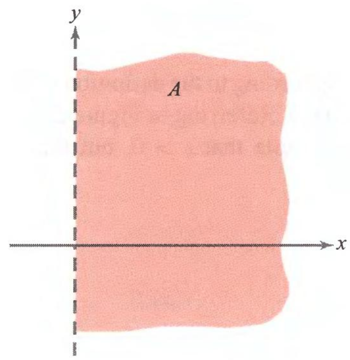


Figure 2.2.4 Show that A is an open set.

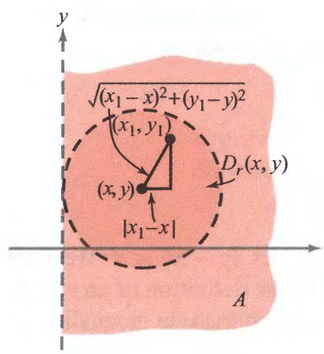


Figure 2.2.5 The construction of a disk about a point in A that is completely enclosed in A.

It is useful to have a special name for an open set containing a given point $\mathbf{x}$, because this idea arises often in the study of limits and continuity. Thus, by a ***neighborhood*** of $\mathbf{x} \in \mathbb{R}^n$ we merely mean an open set U containing the point $\mathbf{x}$. For example, $D_r(\mathbf{x}_0)$ is a neighborhood of $\mathbf{x}_0$ for any $r > 0$. The set A in Example 1 is a neighborhood of the point $\mathbf{x}_0 = (3, -10)$.

Boundary

Let us formally introduce the concept of a boundary point, which we alluded to in Example 1.

> **DEFINITION: Boundary Points** Let $A \subset \mathbb{R}^n$. A point $\mathbf{x} \in \mathbb{R}^n$ is called a ***boundary point*** of A if every neighborhood of $\mathbf{x}$ contains at least one point in A and at least one point not in A.

In this definition, $\mathbf{x}$ itself may or may not be in A; if $\mathbf{x} \in A$, then $\mathbf{x}$ is a boundary point if every neighborhood of $\mathbf{x}$ contains at least one point *not* in A (it already contains a point of A, namely, $\mathbf{x}$). Similarly, if $\mathbf{x}$ is not in A, it is a boundary point if every neighborhood of $\mathbf{x}$ contains at least one point of A.

We shall be particularly interested in boundary points of open sets. By the definition of an open set, no point of an open set A can be a boundary point of A. Thus, *a point* $\mathbf{x}$ *is a boundary point of an open set* A *if and only if* $\mathbf{x}$ *is not in* A *and every neighborhood of* $\mathbf{x}$ *has a nonempty intersection with* A.

This expresses in precise terms the intuitive idea that a boundary point of A is a point just on the "edge" of A. In many examples it is perfectly clear what the boundary points are.

EXAMPLE 2 (a) Let $A = (a, b)$ in $\mathbb{R}$. Then the boundary points of A consist of the points a and b. A consideration of Figure 2.2.6 and the definition will make this clear. [The reader will be asked to prove this in Exercise 20(c).]

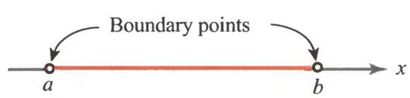


Figure 2.2.6 The boundary points of the interval (a, b).

(b) Let $A = D_r(x_0, y_0)$ be an r-disk about (x_0, y_0) in the plane. The boundary consists of points (x, y) with $(x - x_0)^2 + (y - y_0)^2 = r^2$ (Figure 2.2.7).

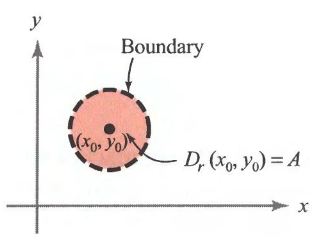


Figure 2.2.7 The boundary of A consists of points on the edge of A.

(c) Let $A = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$. Then the boundary of A consists of all points on the y axis (the student should draw a figure).

(d) Let A be $D_r(\mathbf{x}_0)$ minus the point $\mathbf{x}_0$ (a "punctured" disk about $\mathbf{x}_0$). Then $\mathbf{x}_0$ is a boundary point of A. ▲

Limits

We now turn our attention to the concept of a limit. *Throughout the following discussions the domain of definition of the function* f *will be an open set* A. We are interested in finding the limit of f as $\mathbf{x} \in A$ approaches either a point of A or a boundary point of A.

The reader should appreciate the fact that the limit concept is a basic and useful tool for the analysis of functions; it enables us to study derivatives, and hence maxima and minima, asymptotes, improper integrals, and other important features of functions, as well as being useful for infinite series and sequences. We will present a theory of limits for functions of several variables that includes the theory for functions of one variable as a special case.

In one-variable calculus, the student has encountered the notion of $\lim_{x \to x_0} f(x) = l$ for a function $f\colon A \subset \mathbb{R} \to \mathbb{R}$ from a subset A of the real numbers to the real numbers. Intuitively, this means that as x gets closer and closer to x_0, the values $f(x)$ get closer and closer to (the limiting value) l. To put this intuitive idea on a firm, mathematical foundation, either the "epsilon (ε) and delta (δ) method" or the "neighborhood method" is usually introduced. The same is true for functions of several variables. In what follows we develop the neighborhood approach to limits. The epsilon-delta approach is left for optional study at the end of this section.

DEFINITION: Limit Let $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$, where A is an open set. Let $\mathbf{x}_0$ be in A or be a boundary point of A, and let N be a neighborhood of $\mathbf{b} \in \mathbb{R}^m$. We say f is ***eventually in N as $\mathbf{x}$ approaches*** $\mathbf{x}_0$ if there exists a neighborhood U of $\mathbf{x}_0$ such that $\mathbf{x} \neq \mathbf{x}_0$, $\mathbf{x} \in U$, and $\mathbf{x} \in A$ imply $f(\mathbf{x}) \in N$. [The geometric meaning of this assertion is illustrated in Figure 2.2.8; note that $\mathbf{x}_0$ need not be in the set A, so that $f(\mathbf{x}_0)$ is not necessarily defined.] We say $f(\mathbf{x})$ ***approaches*** $\mathbf{b}$ as $\mathbf{x}$ approaches $\mathbf{x}_0$, or, in symbols,

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{b} \qquad \text{or} \qquad f(\mathbf{x}) \to \mathbf{b} \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_0,$$

when, given *any* neighborhood N of $\mathbf{b}$, f is eventually in N as $\mathbf{x}$ approaches $\mathbf{x}_0$ [that is, "$f(\mathbf{x})$ is close to $\mathbf{b}$ if $\mathbf{x}$ is close to $\mathbf{x}_0$"]. It may be that as $\mathbf{x}$ approaches $\mathbf{x}_0$, the values $f(\mathbf{x})$ do not get close to any particular number. In this case, we say that $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x})$ ***does not exist***.

Henceforth, whenever we consider the notion $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x})$, we shall always assume that $\mathbf{x}_0$ either belongs to some open set on which f is defined or is on the boundary of such a set.

One reason we insist on $\mathbf{x} \neq \mathbf{x}_0$ in the definition of limit will become clear if we remember from one-variable calculus that we want to be able to define the derivative $f'(x_0)$ of a function f at a point x_0 by

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

and this expression is not defined at $x = x_0$.

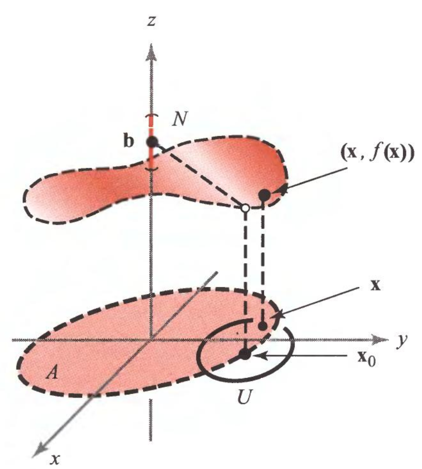


Figure 2.2.8 Limits in terms of neighborhoods; if $\mathbf{x}$ is in U, then $f(\mathbf{x})$ will be in N. (The little open circle denotes that the point does not lie on the graph.) In the figure, $f\colon A = \{(x, y) \mid x^2 + y^2 < 1\} \to \mathbb{R}$. (The dashed line is not in the graph of f.)

EXAMPLE 3 (a) This example illustrates a limit that does not exist. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0. \end{cases}$$

The $\lim_{x\to 0} f(x)$ does not exist, since there are points x_1 arbitrarily close to 0 with $f(x_1) = 1$ and also points x_2 arbitrarily close to 0 with $f(x_2) = -1$; that is, there is no single number that f is close to when x is close to 0 (see Figure 2.2.9). If f is restricted to the domain (0, 1) or $(-1, 0)$, then the limit does exist. Can you say why?

(b) This example illustrates a function whose limit does exist, but whose limiting value does not equal its value at the limiting point. Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

It is true that $\lim_{x\to 0} f(x) = 0$, since for any neighborhood U of 0, $x \in U$ and $x \neq 0$ implies that $f(x) = 0$. One sees from the graph in Figure 2.2.10 that f approaches 0 as $x \to 0$; we do not care that f happens to take on some other value at 0. ▲

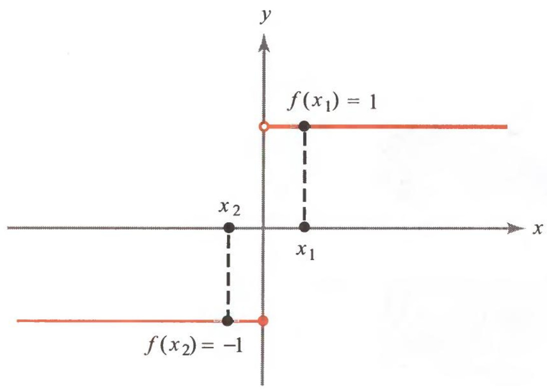


Figure 2.2.9 The limit of this function as $x \to 0$ does not exist.

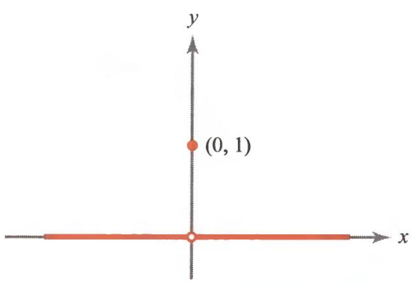


Figure 2.2.10 The limit of this function as $x \to 0$ is zero.

EXAMPLE 4 Use the definition to verify that the "obvious" $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{x} = \mathbf{x}_0$ holds, where $\mathbf{x}$ and $\mathbf{x}_0 \in \mathbb{R}^n$.

SOLUTION Let f be the function defined by $f(\mathbf{x}) = \mathbf{x}$, and let N be any neighborhood of $\mathbf{x}_0$. We must show that $f(\mathbf{x})$ is eventually in N as $\mathbf{x} \to \mathbf{x}_0$. According to the definition, we must find a neighborhood U of $\mathbf{x}_0$ with the property that if $\mathbf{x} \neq \mathbf{x}_0$ and $\mathbf{x} \in U$, then $f(\mathbf{x}) \in N$. Pick $U = N$. If $\mathbf{x} \in U$, then $\mathbf{x} \in N$; because $\mathbf{x} = f(\mathbf{x})$, it follows that $f(\mathbf{x}) \in N$. Thus, we have shown that $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{x} = \mathbf{x}_0$. In a similar way, we have

$$\lim_{(x,y)\to(x_0,y_0)} x = x_0, \qquad \text{etc.} \quad \blacktriangle$$

In what follows, the student may assume, without proof, the validity of limits from one-variable calculus. For example, $\lim_{x\to 1} \sqrt{x} = \sqrt{1} = 1$ and $\lim_{\theta\to 0} \sin\theta = \sin 0 = 0$ may be used.

EXAMPLE 5 (This example demonstrates another case in which the limit cannot simply be "read off" from the function.) Find $\lim_{x\to 1} g(x)$ where

$$g\colon x \mapsto \frac{x-1}{\sqrt{x}-1}.$$

SOLUTION   This function is graphed in Figure 2.2.11(a).

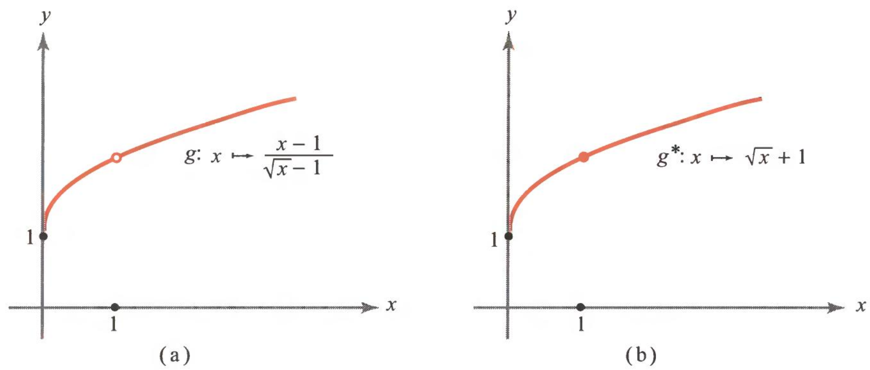


Figure 2.2.11 These graphs are the same except that in part (a), g is undefined at $x = 1$, whereas in part (b), g^* is defined for all $x \geq 0$.

We see that $g(1)$ is not defined, because division by zero is not defined. However, if we multiply the numerator and denominator of $g(x)$ by $\sqrt{x} + 1$, we find that for all x in the domain of g we have

$$g(x) = \frac{x-1}{\sqrt{x}-1} = \sqrt{x} + 1, \qquad x \neq 1.$$

The expression $g^*(x) = \sqrt{x} + 1$ is defined and takes the value 2 at $x = 1$; from one-variable calculus, $g^*(x) \to 2$ as $x \to 1$. But because $g^*(x) = g(x)$ for all $x \geq 0$, $x \neq 1$, we must have as well that $g(x) \to 2$ as $x \to 1$. ▲

We will consider other examples in two variables shortly.

Properties of Limits

To properly speak of *the* limit, we should establish that f can have *at most one* limit as $\mathbf{x} \to \mathbf{x}_0$. This is intuitively clear and we now state it formally. (See the Internet supplement for the proof.)

THEOREM 2: Uniqueness of Limits

$$\text{If} \quad \lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_1 \quad \text{and} \quad \lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_2, \quad \text{then} \quad \mathbf{b}_1 = \mathbf{b}_2.$$

To carry out practical computations with limits, we require some rules for limits, for example, that the limit of a sum is the sum of the limits. These rules are summarized in the following theorem (see the Internet supplement for Chapter 2 for the proof).

THEOREM 3: Properties of Limits Let $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$, $g\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{x}_0$ be in A or be a boundary point of A, $\mathbf{b} \in \mathbb{R}^m$, and $c \in \mathbb{R}$; then

(i) If $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} cf(\mathbf{x}) = c\mathbf{b}$, where $cf\colon A \to \mathbb{R}^m$ is defined by $\mathbf{x} \mapsto c(f(\mathbf{x}))$.

(ii) If $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_1$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = \mathbf{b}_2$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} (f+g)(\mathbf{x}) = \mathbf{b}_1 + \mathbf{b}_2$, where $(f+g)\colon A \to \mathbb{R}^m$ is defined by $\mathbf{x} \mapsto f(\mathbf{x}) + g(\mathbf{x})$.

(iii) If $m = 1$, $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = b_1$, and $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = b_2$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} (fg)(\mathbf{x}) = b_1 b_2$, where $(fg)\colon A \to \mathbb{R}$ is defined by $\mathbf{x} \mapsto f(\mathbf{x})g(\mathbf{x})$.

(iv) If $m = 1$, $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = b \neq 0$, and $f(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in A$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} 1/f(\mathbf{x}) = 1/b$, where $1/f\colon A \to \mathbb{R}$ is defined by $\mathbf{x} \mapsto 1/f(\mathbf{x})$.

(v) If $f(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_m(\mathbf{x}))$ where $f_i\colon A \to \mathbb{R}$, $i = 1, \dots, m$, are the component functions of f, then $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b} = (b_1, \dots, b_m)$ if and only if $\lim_{\mathbf{x}\to\mathbf{x}_0} f_i(\mathbf{x}) = b_i$ for each $i = 1, \dots, m$.

These results ought to be intuitively clear. For instance, rule (ii) says that if $f(\mathbf{x})$ is close to $\mathbf{b}_1$ and $g(\mathbf{x})$ is close to $\mathbf{b}_2$ when $\mathbf{x}$ is close to $\mathbf{x}_0$, then $f(\mathbf{x}) + g(\mathbf{x})$ is close to $\mathbf{b}_1 + \mathbf{b}_2$ when $\mathbf{x}$ is close to $\mathbf{x}_0$. The following example illustrates how this works.

EXAMPLE 6 Let $f\colon \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2 + y^2 + 2$. Compute the limit

$$\lim_{(x,y)\to(0,1)} f(x, y).$$

SOLUTION Here f is the sum of the three functions $(x, y) \mapsto x^2$, $(x, y) \mapsto y^2$, and $(x, y) \mapsto 2$. The limit of a sum is the sum of the limits, and the limit of a product is the product of the limits (Theorem 3). Hence, using the fact that $\lim_{(x,y)\to(x_0,y_0)} x = x_0$ (Example 4), we obtain

$$\lim_{(x,y)\to(x_0,y_0)} x^2 = \left(\lim_{(x,y)\to(x_0,y_0)} x\right)\left(\lim_{(x,y)\to(x_0,y_0)} x\right) = x_0^2$$

and, using the same reasoning, $\lim_{(x,y)\to(x_0,y_0)} y^2 = y_0^2$. Consequently,

$$\lim_{(x,y)\to(0,1)} f(x, y) = 0^2 + 1^2 + 2 = 3. \quad \blacktriangle$$

Continuous Functions

In single-variable calculus we learned that the idea of a continuous function is based on the intuitive notion of a function whose graph is an unbroken curve, that is, a curve that has no *jumps*, or the kind of curve that would be traced by a particle in motion or by a moving pencil point that is not lifted from the paper.

To perform a detailed analysis of functions, we need concepts more precise than this rather vague notion. An example may clarify these ideas. Consider the specific function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -1$ if $x \leq 0$ and $f(x) = 1$ if $x > 0$. The graph of f is shown in Figure 2.2.12(a). [The little open circle denotes the fact that the point (0, 1) does *not* lie on the graph of f]. Clearly, the graph of f is broken at $x = 0$. Consider also the function $g: x \mapsto x^2$. This function is pictured in Figure 2.2.12(b). The graph of g is not broken at any point.

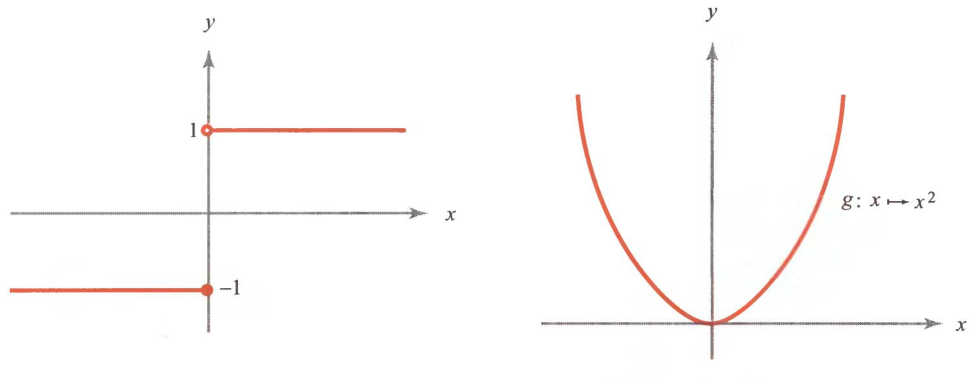


Figure 2.2.12 The function f in part (a) is not continuous, because its value jumps as x crosses 0, whereas the function g in part (b) is continuous.

If one examines examples of functions like f, whose graphs are broken at some point x_0, and functions like g, whose graphs are not broken, one sees that the principal difference between them is that for a function like g, the values of $g(x)$ get closer to $g(x_0)$ as x gets closer and closer to x_0. The same idea works for functions of several variables. But the notion of closer and closer does not suffice as a mathematical definition; thus, we shall formulate these concepts precisely in terms of limits.

Because the condition $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$ means that $f(\mathbf{x})$ is close to $f(\mathbf{x}_0)$ when $\mathbf{x}$ is close to $\mathbf{x}_0$, we see that this limit condition does indeed correspond to the requirement that the graph of f be unbroken (see Figure 2.2.13, where we illustrate the case $f: \mathbb{R} \to \mathbb{R}$). The case of several variables is easiest to visualize if we deal with real-valued functions, say $f: \mathbb{R}^2 \to \mathbb{R}$. In this case, we can visualize f by drawing its graph, which consists of all points (x, y, z) in $\mathbb{R}^3$ with $z = f(x, y)$. The continuity of f thus means that its graph has no "breaks" in it (see Figure 2.2.14).

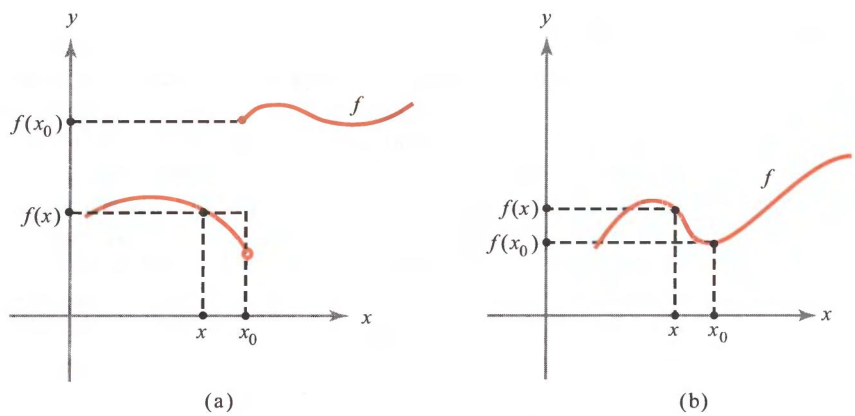


Figure 2.2.13 (a) Discontinuous function for which $\text{limit}_{x \to x_0} f(x)$ does not exist. (b) Continuous function for which this limit exists and equals $f(x_0)$.

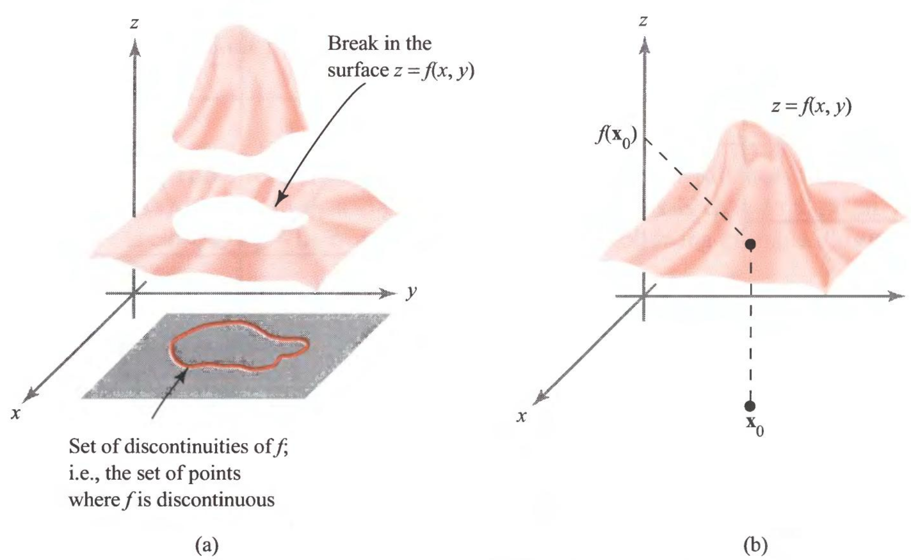


Figure 2.2.14 (a) A discontinuous function of two variables. (b) A continuous function.

DEFINITION: Continuity Let $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a given function with domain A. Let $\mathbf{x}_0 \in A$. We say f is ***continuous*** at $\mathbf{x}_0$ if and only if

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0).$$

If we just say that f is ***continuous***, we shall mean that f is continuous at each point $\mathbf{x}_0$ of A. If f is not continuous at $\mathbf{x}_0$, we say f is ***discontinuous*** at $\mathbf{x}_0$. If f is discontinuous at some point in its domain, we say f is ***discontinuous***.

EXAMPLE 7 Any polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is continuous from $\mathbb{R}$ to $\mathbb{R}$. Indeed, from Theorem 3 and Example 4,

$$\begin{aligned}\lim_{x\to x_0}\left(a_0 + a_1 x + \cdots + a_n x^n\right) &= \lim_{x\to x_0} a_0 + \lim_{x\to x_0} a_1 x + \cdots + \lim_{x\to x_0} a_n x^n \\ &= a_0 + a_1 x_0 + \cdots + a_n x_0^n,\end{aligned}$$

because the limit of a product is the product of the limits, which gives

$$\lim_{x\to x_0} x^n = \left(\lim_{x\to x_0} x\right)^n = x_0^n. \quad ▲$$

EXAMPLE 8 Let $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = xy$. Then f is continuous, because, by the limit theorems and Example 4,

$$\lim_{(x,y)\to(x_0,y_0)} xy = \left(\lim_{(x,y)\to(x_0,y_0)} x\right)\left(\lim_{(x,y)\to(x_0,y_0)} y\right) = x_0 y_0. \quad ▲$$

One can see by the same method that any polynomial $p(x, y)$ [for example, $p(x, y) = 3x^2 - 6xy^2 + y^3$] in x and y is continuous.

EXAMPLE 9 The function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} 1 & \text{if } x \le 0 \text{ or } y \le 0 \\ 0 & \text{otherwise} \end{cases}$$

is not continuous at (0, 0) or at any point on the positive x axis or positive y axis. Indeed, if $(x_0, y_0) = \mathbf{u}$ is such a point (i.e., $x_0 = 0$ and $y_0 \ge 0$, or $y_0 = 0$ and $x_0 \ge 0$) and $\delta > 0$, there are points $(x, y) \in D_\delta(\mathbf{u})$, a neighborhood of $\mathbf{u}$, with $f(x, y) = 1$ and other points $(x, y) \in D_\delta(\mathbf{u})$ with $f(x, y) = 0$. Thus, it is *not* true that $f(x, y) \to f(x_0, y_0) = 1$ as $(x, y) \to (x_0, y_0)$. ▲

To prove that specific functions are continuous, we can avail ourselves of the limit theorems (see Theorem 3 and Example 7). If we transcribe those results in terms of continuity, we are led to the following:

THEOREM 4: Properties of Continuous Functions Suppose that $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$, $g\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$, and let c be a real number.

(i) If f is continuous at $\mathbf{x}_0$, so is cf, where $(cf)(\mathbf{x}) = c[f(\mathbf{x})]$.

(ii) If f and g are continuous at $\mathbf{x}_0$, so is $f + g$, where the sum of f and g is defined by $(f + g)(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x})$.

(iii) If f and g are continuous at $\mathbf{x}_0$ and $m = 1$, then the product function fg defined by $(fg)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ is continuous at $\mathbf{x}_0$.

(iv) If $f\colon A \subset \mathbb{R}^n \to \mathbb{R}$ is continuous at $\mathbf{x}_0$ and nowhere zero on A, then the quotient $1/f$ is continuous at $\mathbf{x}_0$, where $(1/f)(\mathbf{x}) = 1/f(\mathbf{x})$.

(v) If $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ and $f(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_m(\mathbf{x}))$, then f is continuous at $\mathbf{x}_0$ if and only if each of the real-valued functions $f_1, \dots, f_m$ is continuous at $\mathbf{x}_0$.

A variant of (iv) is often used: If $f(\mathbf{x}_0) \neq 0$ and f is continuous, then $f(\mathbf{x}) \neq 0$ in a neighborhood of $\mathbf{x}_0$ and so $1/f$ is defined in that neighborhood, and $1/f$ is continuous at $\mathbf{x}_0$.

EXAMPLE 10 Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (x^2y, (y + x^3)/(1 + x^2))$. Show that f is continuous.

SOLUTION To see this, it is sufficient, by property (v) of Theorem 4, to show that each component is continuous. As we have mentioned, any polynomial in two variables is continuous; thus, the map $(x, y) \mapsto x^2y$ is continuous. Because $1 + x^2$ is continuous and nonzero, by property (iv), we know that $1/(1 + x^2)$ is continuous; hence, $(y + x^3)/(1 + x^2)$ is a product of continuous functions, and by (iii) is continuous. ▲

Similar reasoning applies to examples like the function $\mathbf{c}\colon \mathbb{R} \to \mathbb{R}^3$ given by $\mathbf{c}(t) = (t^2, 1, t^3/(1 + t^2))$ to show they are continuous as well.

Composition

Next we discuss *composition*, another basic operation that can be performed on functions. If g maps A to B and f maps B to C, the ***composition of g with f***, or of f on g, denoted by $f \circ g$, maps A to C by sending $\mathbf{x} \mapsto f(g(\mathbf{x}))$ (see Figure 2.2.15). For example, $\sin(x^2)$ is the composition of $x \mapsto x^2$ with $y \mapsto \sin y$.

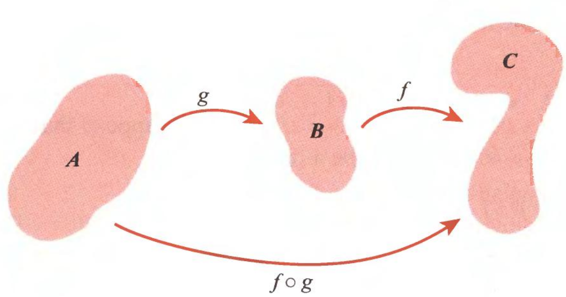


Figure 2.2.15 The composition of f on g.

THEOREM 5: Continuity of Compositions Let $g\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ and let $f\colon B \subset \mathbb{R}^m \to \mathbb{R}^p$. Suppose $g(A) \subset B$, so that $f \circ g$ is defined on A. If g is continuous at $\mathbf{x}_0 \in A$ and f is continuous at $\mathbf{y}_0 = g(\mathbf{x}_0)$, then $f \circ g$ is continuous at $\mathbf{x}_0$.

The intuition behind this is easy; the formal proof in the Internet supplement follows a similar pattern. Intuitively, we must show that as $\mathbf{x}$ gets close to $\mathbf{x}_0$, $f(g(\mathbf{x}))$ gets close to $f(g(\mathbf{x}_0))$. But as $\mathbf{x}$ gets close to $\mathbf{x}_0$, $g(\mathbf{x})$ gets close to $g(\mathbf{x}_0)$ (by continuity of g at $\mathbf{x}_0$); and as $g(\mathbf{x})$ gets close to $g(\mathbf{x}_0)$, $f(g(\mathbf{x}))$ gets close to $f(g(\mathbf{x}_0))$ [by continuity of f at $g(\mathbf{x}_0)$].

EXAMPLE 11 Let $f(x, y, z) = (x^2 + y^2 + z^2)^{30} + \sin z^3$. Show that f is continuous.

SOLUTION Here we can write f as a sum of the two functions $(x^2 + y^2 + z^2)^{30}$ and $\sin z^3$, so it suffices to show that each is continuous. The first is the composite of $(x, y, z) \mapsto (x^2 + y^2 + z^2)$ with $u \mapsto u^{30}$, and the second is the composite of $(x, y, z) \mapsto z^3$ with $u \mapsto \sin u$, and so we have continuity by Theorem 5. ▲

Limits in Terms of ε's and δ's

We now state a theorem (proved in the Internet supplement for Chapter 2) giving a useful formulation of the notion of limit in terms of epsilons and deltas that is often taken as the *definition* of limit. This is, in fact, another way of making precise the intuitive statement that "$f(\mathbf{x})$ is close to $\mathbf{b}$ when $\mathbf{x}$ is close to $\mathbf{x}_0$." To help understand this formulation, the reader should consider it with respect to each of the examples already presented.

THEOREM 6 Let $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbf{x}_0$ be in A or be a boundary point of A. Then $\underset{\mathbf{x}\to\mathbf{x}_0}{\text{limit}}\, f(\mathbf{x}) = \mathbf{b}$ if and only if for every number $\varepsilon > 0$ there is a $\delta > 0$ such that for any $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$, we have $\|f(\mathbf{x}) - \mathbf{b}\| < \varepsilon$ (see Figure 2.2.16).

To illustrate the methodology of the epsilon-delta technique in Theorem 6, we consider the following examples.

EXAMPLE 12 Show that $\underset{(x,y)\to(0,0)}{\text{limit}}\, x = 0$ using the ε-δ method.

SOLUTION Note that if $\delta > 0$, $\|(x, y) - (0, 0)\| = \sqrt{x^2 + y^2} < \delta$ implies $|x - 0| = |x| = \sqrt{x^2} \le \sqrt{x^2 + y^2} < \delta$. Thus, if $\|(x, y) - (0, 0)\| < \delta$, then $|x - 0|$ is also less than δ. Given $\varepsilon > 0$, we are required to find a $\delta > 0$ (generally depending on ε) with the property that $0 < \|(x, y) - (0, 0)\| < \delta$ implies $|x - 0| < \varepsilon$. What are we to pick as our δ? From the preceding calculation, we see that if we choose $\delta = \varepsilon$,

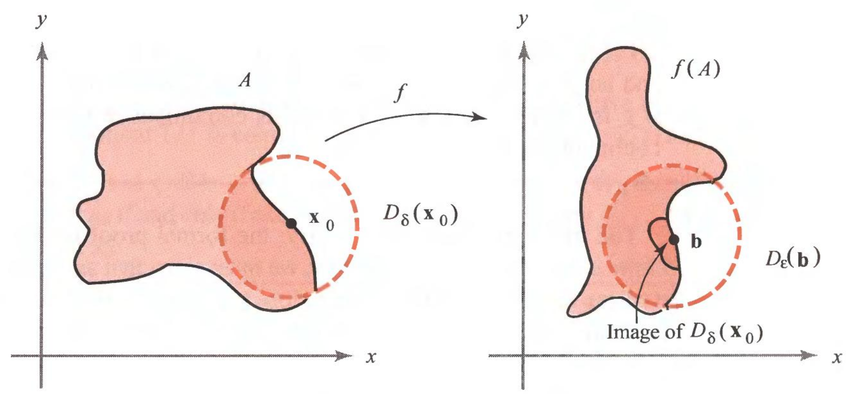


Figure 2.2.16 The geometry of the ε-δ definition of limit.

then $\|(x, y) - (0, 0)\| < \delta$ implies $|x - 0| < \varepsilon$. This shows that $\displaystyle\lim_{(x,y)\to(0,0)} x = 0$. Given $\varepsilon > 0$, we could have also chosen $\delta = \varepsilon/2$ or $\varepsilon/3$, but it suffices to find just one δ satisfying the requirements of the definition of a limit. ▲

EXAMPLE 13 Consider the function

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}.$$

Even though f is not defined at (0, 0), determine whether $f(x, y)$ approaches some number as (x, y) approaches (0, 0).

SOLUTION From one-variable calculus or L'Hôpital's rule we know that

$$\operatorname*{limit}_{\alpha\to 0} \frac{\sin\alpha}{\alpha} = 1.$$

Thus, it is reasonable to guess that

$$\operatorname*{limit}_{\mathbf{v}\to(0,0)} f(\mathbf{v}) = \operatorname*{limit}_{\mathbf{v}\to(0,0)} \frac{\sin\|\mathbf{v}\|^2}{\|\mathbf{v}\|^2} = 1.$$

Indeed, because $\operatorname*{limit}_{\alpha\to 0}(\sin\alpha)/\alpha = 1$, given $\varepsilon > 0$ we are able to find a $\delta > 0$, with $0 < \delta < 1$, such that $0 < |\alpha| < \delta$ implies that $|(\sin\alpha)/\alpha - 1| < \varepsilon$. If $0 < \|\mathbf{v}\| < \delta$, then $0 < \|\mathbf{v}\|^2 < \delta^2 < \delta$, and therefore

$$|f(\mathbf{v}) - 1| = \left|\frac{\sin\|\mathbf{v}\|^2}{\|\mathbf{v}\|^2} - 1\right| < \varepsilon.$$

Thus, $\operatorname*{limit}_{\mathbf{v}\to(0,0)} f(\mathbf{v}) = 1$. If we plot $[\sin(x^2 + y^2)]/(x^2 + y^2)$ on a computer, we get a graph that is indeed well behaved near (0, 0) (Figure 2.2.17). ▲

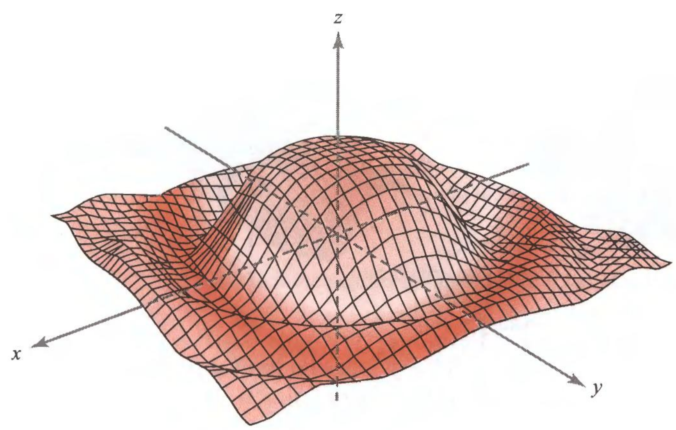


Figure 2.2.17 Graph of the function $f(x, y) = [\sin(x^2 + y^2)]/(x^2 + y^2)$.

EXAMPLE 14 Show that

$$\underset{(x,y)\to(0,0)}{\text{limit}} \frac{x^2}{\sqrt{x^2+y^2}} = 0.$$

SOLUTION We must show that $x^2/\sqrt{x^2+y^2}$ is small when (x, y) is close to the origin. To do this, we use the following inequality:

$$0 \le \frac{x^2}{\sqrt{x^2+y^2}} \le \frac{x^2+y^2}{\sqrt{x^2+y^2}} \qquad (\text{because } y^2 \ge 0)$$
$$= \sqrt{x^2+y^2}.$$

Given $\varepsilon > 0$, choose $\delta = \varepsilon$. Then $\|(x, y) - (0, 0)\| = \|(x, y)\| = \sqrt{x^2+y^2}$, and so $\|(x, y) - (0, 0)\| < \delta$ implies that

$$\left|\frac{x^2}{\sqrt{x^2+y^2}} - 0\right| = \frac{x^2}{\sqrt{x^2+y^2}} \le \sqrt{x^2+y^2} = \|(x, y) - (0, 0)\| < \delta = \varepsilon.$$

Thus, the conditions of Theorem 6 have been fulfilled and the limit is verified. ▲

EXAMPLE 15 (a) Does

$$\underset{(x,y)\to(0,0)}{\text{limit}} x^2/(x^2+y^2)$$

exist? [See Figure 2.2.18(a).]

(b) Prove that [see Figure 2.2.18(b)]

$$\underset{(x,y)\to(0,0)}{\text{limit}} \frac{2x^2y}{x^2+y^2} = 0.$$

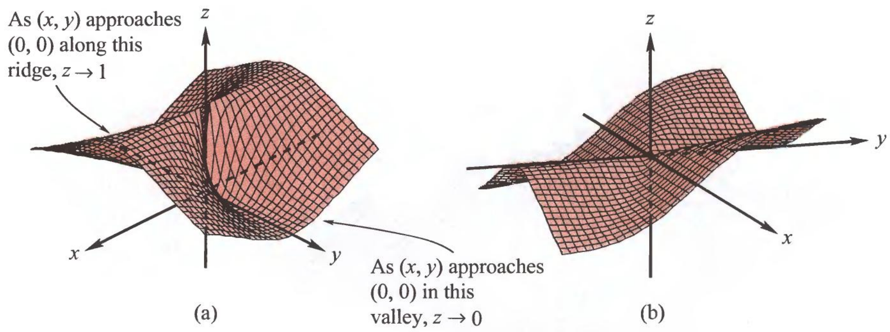


Figure 2.2.18 (a) The function $z = x^2/(x^2 + y^2)$ has no limit at (0, 0). (b) The function $z = (2x^2y)/(x^2 + y^2)$ has limit 0 at (0, 0).

SOLUTION (a) If the limit exists, $x^2/(x^2 + y^2)$ should approach a definite value, say a, as (x, y) gets near (0, 0). In particular, if (x, y) approaches zero along any given path, then $x^2/(x^2 + y^2)$ should approach the limiting value a. If (x, y) approaches (0, 0) along the line $y = 0$, the limiting value is clearly 1 (just set $y = 0$ in the preceding expression to get $x^2/x^2 = 1$). If (x, y) approaches (0, 0) along the line $x = 0$, the limiting value is

$$\lim_{y \to 0} \frac{0^2}{0^2 + y^2} = 0 \neq 1.$$

Hence, $\underset{(x,y)\to(0,0)}{\text{limit}} x^2/(x^2 + y^2)$ does not exist.

(b) Note that

$$\left|\frac{2x^2y}{x^2 + y^2}\right| \leq \left|\frac{2x^2y}{x^2}\right| = 2|y|.$$

Thus, given $\varepsilon > 0$, choose $\delta = \varepsilon/2$; then $0 < \|(x, y) - (0, 0)\| = \sqrt{x^2 + y^2} < \delta$ implies $|y| < \delta$, and thus

$$\left|\frac{2x^2y}{x^2 + y^2} - 0\right| < 2\delta = \varepsilon. \quad \blacktriangle$$

Using the ε-δ notation, we are led to the following reformulation of the definition of continuity.

THEOREM 7 Let $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ be given. Then f is continuous at $\mathbf{x}_0 \in A$ if and only if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\mathbf{x} \in A \quad \text{and} \quad \|\mathbf{x} - \mathbf{x}_0\| < \delta \quad \text{implies} \quad \|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \varepsilon.$$

The proof is almost immediate. Notice that in Theorem 6 we insisted that $0 < \|\mathbf{x} - \mathbf{x}_0\|$, that is, $\mathbf{x} \neq \mathbf{x}_0$. That is *not* imposed here; indeed, the conclusion of Theorem 7 is certainly valid when $\mathbf{x} = \mathbf{x}_0$, and so there is no need to exclude this case. Here we do care about the value of f at $\mathbf{x}_0$; we want f at nearby points to be close to *this* value.

EXERCISES

In the following exercises the reader may assume that the exponential, sine, and cosine functions are continuous and may freely use techniques from one-variable calculus, such as L'Hôpital's rule.

Show that the subsets of the plane in Exercises 1–4 are open:

1. $A = \{(x, y) \mid -1 < x < 1, -1 < y < 1\}$

2. $B = \{(x, y) \mid y > 0\}$

3. $C = \{(x, y) \mid 2 < x^2 + y^2 < 4\}$

4. $D = \{(x, y) \mid x \neq 0 \text{ and } y \neq 0\}$

5. Compute the limits:

(a) $\displaystyle \operatorname*{limit}_{(x,y)\to(0,1)} x^3 y$

(b) $\displaystyle \operatorname*{limit}_{x\to 0} \frac{\cos x - 1}{x^2}$

(c) $\displaystyle \operatorname*{limit}_{h\to 0} \frac{e^h - 1}{h}$.

6. Compute the following limits:

(a) $\displaystyle \operatorname*{limit}_{(x,y)\to(0,1)} e^x y$

(b) $\displaystyle \operatorname*{limit}_{x\to 0} \frac{\sin^2 x}{x}$

(c) $\displaystyle \operatorname*{limit}_{x\to 0} \frac{\sin^2 x}{x^2}$

7. Compute the following limits:

(a) $\displaystyle \operatorname*{limit}_{x\to 3}(x^2 - 3x + 5)$

(b) $\displaystyle \operatorname*{limit}_{x\to 0} \sin x$

(c) $\displaystyle \operatorname*{limit}_{h\to 0} \frac{(x+h)^2 - x^2}{h}$

8. Compute the following limits if they exist:

(a) $\displaystyle \operatorname*{limit}_{(x,y)\to(0,0)} \frac{(x+y)^2 - (x-y)^2}{xy}$

(b) $\displaystyle \operatorname*{limit}_{(x,y)\to(0,0)} \frac{\sin xy}{y}$

(c) $\displaystyle \operatorname*{limit}_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + y^2}$

9. Compute the following limits if they exist:

(a) $\displaystyle \operatorname*{limit}_{(x,y)\to(0,0)} \frac{e^{xy} - 1}{y}$

(b) $\displaystyle \operatorname*{limit}_{(x,y)\to(0,0)} \frac{\cos(xy) - 1}{x^2 y^2}$

(c) $\displaystyle \operatorname*{limit}_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2 + 2}$

10. Compute the following limits, if they exist:

(a) $\lim_{(x,y)\to(0,0)} \dfrac{e^{xy}}{x+1}$

(b) $\lim_{(x,y)\to(0,0)} \dfrac{\cos x - 1 - (x^2/2)}{x^4 + y^4}$

(c) $\lim_{(x,y)\to(0,0)} \dfrac{(x-y)^2}{x^2+y^2}$

11. Compute the following limits if they exist:

(a) $\lim_{(x,y)\to(0,0)} \dfrac{\sin xy}{xy}$

(b) $\lim_{(x,y,z)\to(0,0,0)} \dfrac{\sin(xyz)}{xyz}$

(c) $\lim_{(x,y,z)\to(0,0,0)} f(x,y,z)$, where $f(x,y,z) = (x^2+3y^2)/(x+1)$.

12. Compute the following limits if they exist:

(a) $\lim_{x\to 0} \dfrac{\sin 2x - 2x}{x^3}$

(b) $\lim_{(x,y)\to(0,0)} \dfrac{\sin 2x - 2x + y}{x^3 + y}$

(c) $\lim_{(x,y,z)\to(0,0,0)} \dfrac{2x^2 y \cos z}{x^2+y^2}$

13. Compute $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$, if it exists, for the following cases:

(a) $f\colon \mathbb{R}\to\mathbb{R}, x\mapsto |x|, x_0 = 1$

(b) $f\colon \mathbb{R}^n\to\mathbb{R}, \mathbf{x}\mapsto \|\mathbf{x}\|$, arbitrary $\mathbf{x}_0$

(c) $f\colon \mathbb{R}\to\mathbb{R}^2, x\mapsto (x^2, e^x), x_0 = 1$

(d) $f\colon \mathbb{R}^2\backslash\{(0,0)\}\to\mathbb{R}^2, (x,y)\mapsto (\sin(x-y), e^{x(y+1)} - x - 1)/\|(x,y)\|, \mathbf{x}_0 = (0,0)$.

14. Let $A\subset\mathbb{R}^2$ be the open unit disk $D_1(0,0)$ with the point $\mathbf{x}_0 = (1,0)$ added, and let $f\colon A\to\mathbb{R}, \mathbf{x}\mapsto f(\mathbf{x})$ be the constant function $f(\mathbf{x}) = 1$. Show that $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = 1$.

15. If $f\colon\mathbb{R}^n\to\mathbb{R}$ and $g\colon\mathbb{R}^n\to\mathbb{R}$ are continuous, show that the functions

$$f^2 g\colon \mathbb{R}^n\to\mathbb{R}, \mathbf{x}\mapsto [f(\mathbf{x})]^2 g(\mathbf{x})$$

and

$$f^2 + g\colon \mathbb{R}^n\to\mathbb{R}, x\mapsto [f(\mathbf{x})]^2 + g(\mathbf{x})$$

are continuous.

16. (a) Show that $f\colon\mathbb{R}\to\mathbb{R}, x\mapsto (1-x)^8 + \cos(1+x^3)$ is continuous.
(b) Show that the map $f\colon\mathbb{R}\to\mathbb{R}, x\mapsto x^2 e^x/(2-\sin x)$ is continuous.

17. (a) Can $[\sin(x+y)]/(x+y)$ be made continuous by suitably defining it at $(0,0)$?
(b) Can $xy/(x^2+y^2)$ be made continuous by suitably defining it at $(0,0)$?
(c) Prove that $f\colon\mathbb{R}^2\to\mathbb{R}, (x,y)\mapsto ye^x + \sin x + (xy)^4$ is continuous.

18. Using either ε's and δ's or spherical coordinates, show that

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = 0.$$

19. Use the ε-δ formulation of limits to prove that $x^2 \to 4$ as $x \to 2$. Give another proof using Theorem 3.

20. (a) Prove that for $\mathbf{x} \in \mathbb{R}^n$ and $s < t$, $D_s(\mathbf{x}) \subset D_t(\mathbf{x})$.
(b) Prove that if U and V are neighborhoods of $\mathbf{x} \in \mathbb{R}^n$, then so are $U \cap V$ and $U \cup V$.
(c) Prove that the boundary points of an open interval $(a, b) \subset \mathbb{R}$ are the points a and b.

21. Suppose $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{y}$. Show that there is a continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ with $f(\mathbf{x}) = 1$, $f(\mathbf{y}) = 0$, and $0 \leq f(\mathbf{z}) \leq 1$ for every $\mathbf{z}$ in $\mathbb{R}^n$.

22. Let $f\colon A \subset \mathbb{R}^n \to \mathbb{R}$ be given and let $\mathbf{x}_0$ be a boundary point of A. We say that $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \infty$ if for every $N > 0$ there is a $\delta > 0$ such that $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$ and $\mathbf{x} \in A$ implies $f(\mathbf{x}) > N$.

(a) Prove that $\lim_{x\to 1} (x-1)^{-2} = \infty$.
(b) Prove that $\lim_{x\to 0} 1/|x| = \infty$. Is it true that $\lim_{x\to 0} 1/x = \infty$?
(c) Prove that $\lim_{(x,y)\to(0,0)} 1/(x^2+y^2) = \infty$.

23. Let $b \in \mathbb{R}$ and $f\colon \mathbb{R}\backslash[b] \to \mathbb{R}$ be a function. We write $\lim_{x\to b-} f(x) = L$ and say that L is the ***left-hand limit*** of f at b, if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $x < b$ and $0 < |x - b| < \delta$ implies $|f(x) - L| < \varepsilon$.

(a) Formulate a definition of ***right-hand limit***, or $\lim_{x\to b+} f(x)$.
(b) Find $\lim_{x\to 0-} 1/(1+e^{1/x})$ and $\lim_{x\to 0+} 1/(1+e^{1/x})$.
(c) Sketch the graph of $1/(1+e^{1/x})$.

24. Show that f is continuous at $\mathbf{x}_0$ if and only if

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \|f(\mathbf{x}) - f(\mathbf{x}_0)\| = 0.$$

25. Let $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ satisfy $\|f(\mathbf{x}) - f(\mathbf{y})\| \leq K\|\mathbf{x} - \mathbf{y}\|^\alpha$ for all $\mathbf{x}$ and $\mathbf{y}$ in A for positive constants K and α. Show that f is continuous. (Such functions are called ***Hölder-continuous*** or, if $\alpha = 1$, ***Lipschitz-continuous***.)

26. Show that $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous at all points if and only if the inverse image of every open set is open.

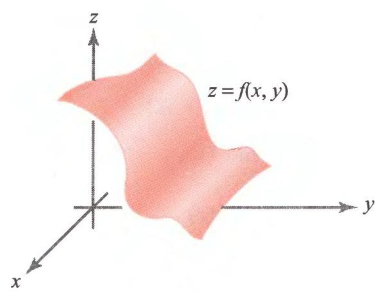
27. (a) Find a specific number $\delta > 0$ such that if $|a| < \delta$, then $|a^3 + 3a^2 + a| < 1/100$.
(b) Find a specific number $\delta > 0$ such that if $x^2 + y^2 < \delta^2$, then

$$|x^2 + y^2 + 3xy + 180xy^5| < 1/10{,}000.$$

2.3 Differentiation

In Section 2.1 we considered a few methods for graphing functions. By these methods alone it may be impossible to compute enough information to grasp even the general features of a complicated function. From elementary calculus we know that the idea of the derivative can greatly aid us in this task; for example, it enables us to locate maxima and minima and to compute rates of change. The derivative also has many applications beyond this, as the student surely has discovered in elementary calculus.

Intuitively, we know from our work in Section 2.2 that a continuous function is one that has no "breaks" in its graph. A differentiable function from $\mathbb{R}^2$ to $\mathbb{R}$ ought to be such that not only are there no breaks in its graph, but there is a well-defined plane tangent to the graph at each point. Thus, there must not be any sharp folds, corners, or peaks in the graph (see Figure 2.3.1). In other words, the graph must be *smooth*.



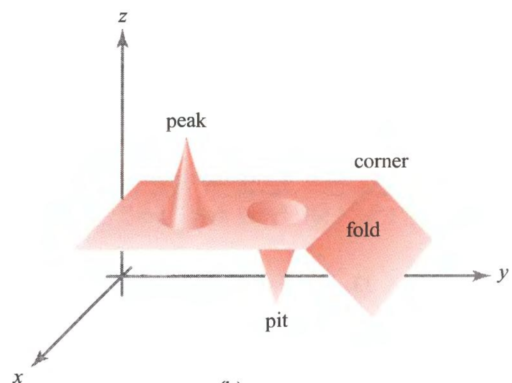


Figure 2.3.1 (a) A smooth graph and (b) a nonsmooth one.

Partial Derivatives

To make these ideas precise, we need a sound definition of what we mean by the phrase "$f(x_1, \ldots, x_n)$ is differentiable at $\mathbf{x} = (x_1, \ldots, x_n)$." Actually, this definition is not quite as simple as one might think. Toward this end, however, let us introduce the notion of the *partial derivative*. This notion relies only on our knowledge of one-variable calculus. (A quick review of the definition of the derivative in a one-variable calculus text might be advisable at this point.)

DEFINITION: Partial Derivatives Let $U \subset \mathbb{R}^n$ be an open set and suppose $f: U \subset \mathbb{R}^n \to \mathbb{R}$ is a real-valued function. Then $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$, the ***partial derivatives*** of f with respect to the first, second, ..., nth variable, are the real-valued functions of n variables, which, at the point $(x_1, \ldots, x_n) = \mathbf{x}$, are defined by

$$\frac{\partial f}{\partial x_j}(x_1, \ldots, x_n) = \lim_{h\to 0} \frac{f(x_1, x_2, \ldots, x_j + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

$$= \lim_{h\to 0} \frac{f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x})}{h}$$

if the limits exist, where $1 \le j \le n$ and $\mathbf{e}_j$ is the jth standard basis vector defined by $\mathbf{e}_j = (0, \ldots, 1, \ldots, 0)$, with 1 in the jth slot (see Section 1.5). The domain of the function $\partial f/\partial x_j$ is the set of $\mathbf{x} \in \mathbb{R}^n$ for which the limit exists.

In other words, $\partial f/\partial x_j$ is just the derivative of f with respect to the variable x_j, with the other variables held fixed. If $f: \mathbb{R}^3 \to \mathbb{R}$, we shall often use the notation $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$ in place of $\partial f/\partial x_1$, $\partial f/\partial x_2$, $\partial f/\partial x_3$. If $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$, then we can write

$$f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)),$$

so that we can speak of the partial derivatives of each component; for example, $\partial f_m/\partial x_n$ is the partial derivative of the mth *component* with respect to x_n, the nth variable.

EXAMPLE 1 If $f(x, y) = x^2y + y^3$, find $\partial f/\partial x$ and $\partial f/\partial y$.

SOLUTION To find $\partial f/\partial x$ we hold y constant (think of it as some number, say 1) and differentiate only with respect to x; this yields

$$\frac{\partial f}{\partial x} = \frac{d(x^2y + y^3)}{dx} = 2xy.$$

Similarly, to find $\partial f/\partial y$ we hold x constant and differentiate only with respect to y:

$$\frac{\partial f}{\partial y} = \frac{d(x^2y + y^3)}{dy} = x^2 + 3y^2. \quad \blacktriangle$$

To indicate that a partial derivative is to be evaluated at a particular point, for example, at (x_0, y_0), we write

$$\frac{\partial f}{\partial x}(x_0, y_0) \qquad \text{or} \qquad \left.\frac{\partial f}{\partial x}\right|_{x=x_0, y=y_0} \qquad \text{or} \qquad \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}.$$

When we write $z = f(x, y)$ for the dependent variable, we sometimes write $\partial z/\partial x$ for $\partial f/\partial x$. Strictly speaking, this is an abuse of notation, but it is common practice to use these two notations interchangeably.

EXAMPLE 2 If $z = \cos xy + x \cos y = f(x, y)$, find the two partial derivatives $(\partial z/\partial x)(x_0, y_0)$ and $(\partial z/\partial y)(x_0, y_0)$.

SOLUTION First we fix y_0 and differentiate with respect to x, giving

$$\begin{aligned} \frac{\partial z}{\partial x}(x_0, y_0) &= \left.\frac{d(\cos xy_0 + x \cos y_0)}{dx}\right|_{x=x_0} \\ &= (-y_0 \sin xy_0 + \cos y_0)|_{x=x_0} \\ &= -y_0 \sin x_0y_0 + \cos y_0. \end{aligned}$$

Similarly, we fix x_0 and differentiate with respect to y to obtain

$$\begin{aligned}\frac{\partial z}{\partial y}(x_0, y_0) &= \left.\frac{d(\cos x_0 y + x_0 \cos y)}{dy}\right|_{y=y_0}\\ &= (-x_0 \sin x_0 y - x_0 \sin y)|_{y=y_0}\\ &= -x_0 \sin x_0 y_0 - x_0 \sin y_0. \quad \blacktriangle\end{aligned}$$

EXAMPLE 3 Find $\partial f/\partial x$ if $f(x, y) = xy/\sqrt{x^2 + y^2}$.

SOLUTION By the quotient rule,

$$\frac{\partial f}{\partial x} = \frac{y\sqrt{x^2+y^2} - xy(x/\sqrt{x^2+y^2})}{x^2+y^2} = \frac{y(x^2+y^2) - x^2 y}{(x^2+y^2)^{3/2}} = \frac{y^3}{(x^2+y^2)^{3/2}}. \quad \blacktriangle$$

A definition of differentiability that requires only the existence of partial derivatives turns out to be insufficient. Many standard results, such as the chain rule for functions of several variables would not follow, as Example 4 shows. Below, we shall see how to rectify this situation.

EXAMPLE 4 Let $f(x, y) = x^{1/3} y^{1/3}$. By definition,

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0 - 0}{h} = 0,$$

and, similarly, $(\partial f/\partial y)(0, 0) = 0$ (these are not indeterminate forms!). It is necessary to use the original definition of partial derivatives, because the functions $x^{1/3}$ and $y^{1/3}$ are not themselves differentiable at 0. Suppose we restrict f to the line $y = x$ to get $f(x, x) = x^{2/3}$ (see Figure 2.3.2). We can view the substitution $y = x$ as the composition $f \circ g$ of the function $g\colon \mathbb{R} \to \mathbb{R}^2$, defined by $g(x) = (x, x)$, and $f\colon \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x, y) = x^{1/3} y^{1/3}$.

Thus, the composite $f \circ g$ is given by $(f \circ g)(x) = x^{2/3}$. Each component of g is differentiable in x, and f has partial derivatives at $(0, 0)$, but *$f \circ g$ is not differentiable* at $x = 0$, in the sense of one-variable calculus. In other words, *the composition of f with g is not differentiable* in contrast to the calculus of functions of one variable, where the composition of differentiable functions *is* differentiable. Below, we shall give a definition of differentiability that has the pleasant consequence that the composition of differentiable functions *is* differentiable.

There is another reason for being dissatisfied with the mere existence of partial derivatives of $f(x, y) = x^{1/3} y^{1/3}$: There is no plane tangent, in any reasonable sense, to the graph at $(0, 0)$. The xy plane is tangent to the graph along the x and y axes because f has slope zero at $(0, 0)$ along these axes; that is, $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$

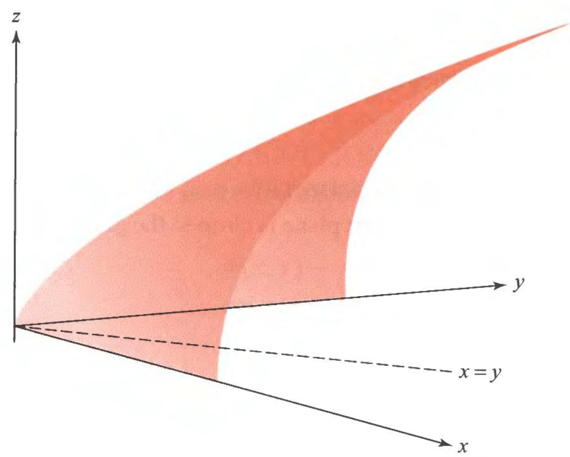


Figure 2.3.2 The portion of the graph of $x^{1/3}y^{1/3}$ in the first quadrant.

at (0, 0). Thus, if there is a tangent plane, it must be the xy plane. However, as is evident from Figure 2.3.2, the xy plane is not tangent to the graph in other directions, because the graph has a severe crinkle, and so the xy plane cannot be said to be tangent to the graph of f. ▲

The Linear Approximation

To "motivate" our definition of differentiability, let us compute what the equation of the plane tangent to the graph of $f: \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$ at (x_0, y_0) ought to be if f is smooth enough. In $\mathbb{R}^3$, a nonvertical plane has an equation of the form

$$z = ax + by + c.$$

If it is to be the plane tangent to the graph of f, the slopes along the x and y axes must be equal to $\partial f/\partial x$ and $\partial f/\partial y$, the rates of change of f with respect to x and y. Thus, $a = \partial f/\partial x$, $b = \partial f/\partial y$ [evaluated at (x_0, y_0)]. Finally, we may determine the constant c from the fact that $z = f(x_0, y_0)$ when $x = x_0$, $y = y_0$. Thus, we get the ***linear approximation***:

$$z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0), \tag{1}$$

which should be the equation of the plane tangent to the graph of f at (x_0, y_0), if f is "smooth enough" (see Figure 2.3.3).

Our definition of differentiability will mean in effect that the plane defined by the linear approximation (1) is a "good" approximation of f near (x_0, y_0). To get an idea of what one might mean by a good approximation, let us return for a moment to

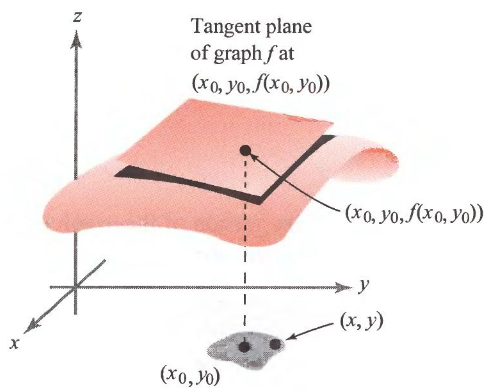


Figure 2.3.3 For points (x, y) near (x_0, y_0), the graph of the tangent plane is close to the graph of f.

one-variable calculus. If f is differentiable at a point x_0, then we know that

$$\operatorname*{limit}_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0).$$

Let $x = x_0 + \Delta x$ and rewrite this as

$$\operatorname*{limit}_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).$$

Using the trivial limit $\operatorname*{limit}_{x \to x_0} f'(x_0) = f'(x_0)$, we can rewrite the preceding equation as

$$\operatorname*{limit}_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \operatorname*{limit}_{x \to x_0} f'(x_0);$$

that is,

$$\operatorname*{limit}_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right] = 0;$$

that is,

$$\operatorname*{limit}_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0.$$

Thus, the tangent line l through $(x_0, f(x_0))$ with slope $f'(x_0)$ is close to f in the sense that the difference between $f(x)$ and $l(x) = f(x_0) + f'(x_0)(x - x_0)$, the equation of the tangent line goes to zero *even* when divided by $x - x_0$ as x goes to x_0. This is the notion of a "good approximation" that we will adapt to functions of several variables, with the tangent line replaced by the tangent plane [see equation (1), given earlier].

Differentiability for Functions of Two Variables

Using the linear approximation, we are ready to define the notion of differentiability.

DEFINITION: Differentiable: Two Variables Let $f\colon \mathbb{R}^2 \to \mathbb{R}$. We say f is ***differentiable*** at (x_0, y_0), if $\partial f/\partial x$ and $\partial f/\partial y$ exist at (x_0, y_0) and if

$$\frac{f(x, y) - f(x_0, y_0) - \left[\dfrac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) - \left[\dfrac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)}{\|(x, y) - (x_0, y_0)\|} \to 0 \tag{2}$$

as $(x, y) \to (x_0, y_0)$. This equation expresses what we mean by saying that

$$f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)$$

is a ***good approximation*** to the function f.

It is not always easy to use this definition to see whether f is differentiable, but it will be easy to use another criterion, given shortly in Theorem 9.

Tangent Plane

We have used the informal notion of the plane tangent to the graph of a function to motivate our definition of differentiability. Now we are ready to adopt a formal definition of the tangent plane.

DEFINITION: Tangent Plane Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $\mathbf{x}_0 = (x_0, y_0)$. The plane in $\mathbb{R}^3$ defined by the equation

$$z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0),$$

is called the ***tangent plane*** of the graph of f at the point (x_0, y_0).

EXAMPLE 5 Compute the plane tangent to the graph of $z = x^2 + y^4 + e^{xy}$ at the point $(1, 0, 2)$.

SOLUTION Use formula (1), with $x_0 = 1$, $y_0 = 0$, and $z_0 = f(x_0, y_0) = 2$. The partial derivatives are

$$\frac{\partial z}{\partial x} = 2x + ye^{xy} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 4y^3 + xe^{xy}.$$

At (1, 0, 2), these partial derivatives are 2 and 1, respectively. Thus, by formula (1), the tangent plane is

$$z = 2(x-1) + 1(y-0) + 2, \qquad \text{that is,} \qquad z = 2x + y. \quad \blacktriangle$$

Let us write $\mathbf{D}f(x_0, y_0)$ for the row matrix

$$\left[\frac{\partial f}{\partial x}(x_0, y_0) \quad \frac{\partial f}{\partial y}(x_0, y_0)\right],$$

so that the definition of differentiability asserts that

$$\begin{aligned} f(x_0, y_0) &+ \mathbf{D}f(x_0, y_0)\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \\ &= f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0) \end{aligned} \tag{3}$$

is our good approximation to f near (x_0, y_0). As earlier, "good" is taken in the sense that expression (3) differs from $f(x, y)$ by something small times $\sqrt{(x - x_0)^2 + (y - y_0)^2}$. We say that expression (3) is the ***best linear approximation*** to f near (x_0, y_0).

Differentiability: The General Case

Now we are ready to give a definition of differentiability for maps f of $\mathbb{R}^n$ to $\mathbb{R}^m$, using the preceding discussion as motivation. The derivative $\mathbf{D}f(\mathbf{x}_0)$ of $f = (f_1, \ldots, f_m)$ at a point $\mathbf{x}_0$ is a matrix $\mathbf{T}$ whose elements are $t_{ij} = \partial f_i / \partial x_j$ evaluated at $\mathbf{x}_0$.[2]

DEFINITION: Differentiable, *n* Variables, *m* Functions Let U be an open set in $\mathbb{R}^n$ and let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a given function. We say that f is ***differentiable*** at $\mathbf{x}_0 \in U$ if the partial derivatives of f exist at $\mathbf{x}_0$ and if

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\| f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{T}(\mathbf{x} - \mathbf{x}_0) \|}{\| \mathbf{x} - \mathbf{x}_0 \|} = 0, \tag{4}$$

where $\mathbf{T} = \mathbf{D}f(\mathbf{x}_0)$ is the $m \times n$ matrix with matrix elements $\partial f_i / \partial x_j$ evaluated at $\mathbf{x}_0$ and $\mathbf{T}(\mathbf{x} - \mathbf{x}_0)$ means the product of $\mathbf{T}$ with $\mathbf{x} - \mathbf{x}_0$ (regarded as a column matrix). We call $\mathbf{T}$ the ***derivative*** of f at $\mathbf{x}_0$.

[2]It turns out that we need to postulate the existence of only *some* matrix giving the best linear approximation near $\mathbf{x}_0 \in \mathbb{R}^n$, because in fact this matrix is *necessarily* the matrix whose ijth entry is $\partial f_i / \partial x_j$ (see the Internet supplement for Chapter 2).

We shall always denote the derivative $\mathbf{T}$ of f at $\mathbf{x}_0$ by $\mathbf{D}f(\mathbf{x}_0)$, although in some books it is denoted $df(\mathbf{x}_0)$ and referred to as the ***differential*** of f. In the case where $m = 1$, the matrix $\mathbf{T}$ is just the row matrix

$$\begin{bmatrix} \dfrac{\partial f}{\partial x_1}(\mathbf{x}_0) & \cdots & \dfrac{\partial f}{\partial x_n}(\mathbf{x}_0) \end{bmatrix}.$$

(Sometimes, when there is danger of confusion, we separate the entries by commas.) Furthermore, setting $n = 2$ and putting the result back into equation (4), we see that conditions (2) and (4) do agree. Thus, if we let $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$, a real-valued function f of n variables is differentiable at a point $\mathbf{x}_0$ if

$$\lim_{\mathbf{h}\to 0} \frac{1}{\|\mathbf{h}\|}\left| f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\mathbf{x}_0)h_j \right| = 0,$$

because

$$\mathbf{T}\mathbf{h} = \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(\mathbf{x}_0).$$

For the general case of f mapping a subset of $\mathbb{R}^n$ to $\mathbb{R}^m$, the derivative is the $m \times n$ matrix given by

$$\mathbf{D}f(\mathbf{x}_0) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix},$$

where $\partial f_i/\partial x_j$ is evaluated at $\mathbf{x}_0$. The matrix $\mathbf{D}f(\mathbf{x}_0)$ is, appropriately, called the ***matrix of partial derivatives of f at*** $\mathbf{x}_0$.

EXAMPLE 6 Calculate the matrices of partial derivatives for these functions.

(a) $f(x, y) = (e^{x+y} + y, y^2x)$

(b) $f(x, y) = (x^2 + \cos y, ye^x)$

(c) $f(x, y, z) = (ze^x, -ye^z)$

SOLUTION

(a) Here $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f_1(x, y) = e^{x+y} + y$ and $f_2(x, y) = y^2x$. Hence, $\mathbf{D}f(x, y)$ is the 2×2 matrix

$$\mathbf{D}f(x, y) = \begin{bmatrix} e^{x+y} & e^{x+y} + 1 \\ y^2 & 2xy \end{bmatrix}.$$

(b) We have

$$\mathbf{D}f(x,y) = \begin{bmatrix} 2x & -\sin y \\ ye^x & e^x \end{bmatrix}.$$

(c) In this case,

$$\mathbf{D}f(x,y,z) = \begin{bmatrix} ze^x & 0 & e^x \\ 0 & -e^z & -ye^z \end{bmatrix}. \quad \blacktriangle$$

Gradients

For real-valued functions we use special terminology for the derivative.

DEFINITION: Gradient Consider the special case $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$. Here $\mathbf{D}f(\mathbf{x})$ is a $1 \times n$ matrix:

$$\mathbf{D}f(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \end{bmatrix}.$$

We can form the corresponding vector $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$, called the ***gradient*** of f and denoted by ∇f or grad f.

From the definition, we see that for $f\colon \mathbb{R}^3 \to \mathbb{R}$,

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k},$$

while for $f\colon \mathbb{R}^2 \to \mathbb{R}$,

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

The geometric significance of the gradient will be discussed in Section 2.6. In terms of inner products, we can write the derivative of f as

$$\mathbf{D}f(\mathbf{x})(\mathbf{h}) = \nabla f(\mathbf{x}) \cdot \mathbf{h}.$$

EXAMPLE 7 Let $f\colon \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = xe^y$. Then

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (e^y, xe^y, 0). \quad \blacktriangle$$

EXAMPLE 8 If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is given by $(x, y) \mapsto e^{xy} + \sin xy$, then

$$\begin{aligned}\nabla f(x, y) &= (ye^{xy} + y\cos xy)\mathbf{i} + (xe^{xy} + x\cos xy)\mathbf{j}\\ &= (e^{xy} + \cos xy)(y\mathbf{i} + x\mathbf{j}).\end{aligned}$$ ▲

In one-variable calculus it is shown that if f is differentiable, then f is continuous. We will state in Theorem 8 that this is also true for differentiable functions of several variables. As we know, there are plenty of functions of one variable that are continuous but not differentiable, such as $f(x) = |x|$. Before stating the result, let us give an example of a function of two variables whose *partial derivatives exist at a point, but which is not continuous at that point.*

EXAMPLE 9 Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or if } y = 0\\ 0 & \text{otherwise.}\end{cases}$$

Because f is constant on the x and y axes, where it equals 1,

$$\frac{\partial f}{\partial x}(0, 0) = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(0, 0) = 0.$$

But f is not continuous at (0, 0), because $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist. ▲

Some Basic Theorems

The first of these basic theorems relates differentiability and continuity.

THEOREM 8 Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\mathbf{x}_0 \in U$. Then f is continuous at $\mathbf{x}_0$.

This result is very reasonable, because "differentiability" means that there is enough smoothness to have a tangent plane, which is stronger than just being continuous. Consult the Internet supplement for Chapter 2 for the formal proof.

As we have seen, it is usually easy to tell when the partial derivatives of a function exist using what we know from one-variable calculus. However, the definition of differentiability looks somewhat complicated, and the required approximation condition in equation (4) may seem, and sometimes is, difficult to verify. Fortunately, there is a simple criterion, given in the following theorem, that tells us when a function is differentiable.

THEOREM 9 Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$. Suppose the partial derivatives $\partial f_i/\partial x_j$ of f all exist and are continuous in a neighborhood of a point $\mathbf{x} \in U$. Then f is differentiable at $\mathbf{x}$.

We give the proof in the Internet supplement for Chapter 2. Notice the following hierarchy:

	Theorem 9		Definition of derivative	
	$\downarrow$		$\downarrow$	
Continuous partials $\Rightarrow$	Differentiable	$\Rightarrow$	Partials exist	

Each converse statement, obtained by reversing an implication, is invalid. [For a counterexample to the converse of the first implication, use $f(x) = x^2 \sin(1/x)$, $f(0) = 0$; for the second, see Example 1 in the Internet supplement for Chapter 2 or use Example 4 in this section.]

A function whose partial derivatives exist and are continuous is said to be of ***class*** C^1. Thus, Theorem 9 says that *any C^1 function is differentiable.*

EXAMPLE 10 Let

$$f(x, y) = \frac{\cos x + e^{xy}}{x^2 + y^2}.$$

Show that f is differentiable at all points $(x, y) \neq (0, 0)$.

SOLUTION Observe that the partial derivatives

$$\frac{\partial f}{\partial x} = \frac{(x^2 + y^2)(ye^{xy} - \sin x) - 2x(\cos x + e^{xy})}{(x^2 + y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x^2 + y^2)xe^{xy} - 2y(\cos x + e^{xy})}{(x^2 + y^2)^2}$$

are continuous except when $x = 0$ and $y = 0$ (by the results in Section 2.2). Thus, f is differentiable by Theorem 9. ▲

In the Internet supplement we show that $f(x, y) = xy/\sqrt{x^2 + y^2}$ [with $f(0, 0) = 0$] is continuous, has partial derivatives at (0, 0), yet is *not* differentiable there. See Figure 2.3.4. By Theorem 9, its partial derivatives cannot be continuous at (0, 0).

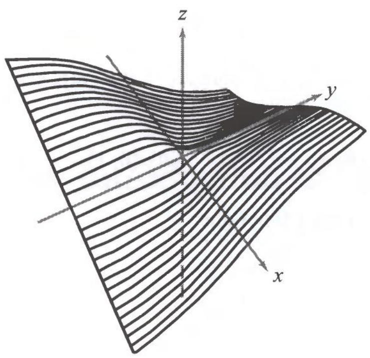


Figure 2.3.4 This function is not differentiable at (0, 0), because it is "crinkled."

EXERCISES

1. Find $\partial f/\partial x$, $\partial f/\partial y$ if

(a) $f(x, y) = xy$
(b) $f(x, y) = e^{xy}$
(c) $f(x, y) = x \cos x \cos y$
(d) $f(x, y) = (x^2 + y^2) \log (x^2 + y^2)$

2. Evaluate the partial derivatives $\partial z/\partial x$, $\partial z/\partial y$ for the given function at the indicated points.

(a) $z = \sqrt{a^2 - x^2 - y^2}$; $(0, 0)$, $(a/2, a/2)$
(b) $z = \log \sqrt{1 + xy}$; $(1, 2)$, $(0, 0)$
(c) $z = e^{ax} \cos (bx + y)$; $(2\pi/b, 0)$

3. In each case following, find the partial derivatives $\partial w/\partial x$, $\partial w/\partial y$.

(a) $w = xe^{x^2+y^2}$
(b) $w = \dfrac{x^2 + y^2}{x^2 - y^2}$
(c) $w = e^{xy} \log (x^2 + y^2)$
(d) $w = x/y$
(e) $w = \cos (ye^{xy}) \sin x$

4. Show that each of the following functions is differentiable at each point in its domain. Decide which of the functions are C^1.

(a) $f(x, y) = \dfrac{2xy}{(x^2 + y^2)^2}$
(b) $f(x, y) = \dfrac{x}{y} + \dfrac{y}{x}$
(c) $f(r, \theta) = \frac{1}{2} r \sin 2\theta$, $r > 0$
(d) $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$
(e) $f(x, y) = \dfrac{x^2 y}{x^4 + y^2}$

5. Find the equation of the plane tangent to the surface $z = x^2 + y^3$ at $(3, 1, 10)$.

6. Using the respective functions in Exercise 1, compute the plane tangent to the graphs at the indicated points.

(a) $(0, 0)$ (b) $(0, 1)$ (c) $(0, \pi)$ (d) $(0, 1)$

7. Compute the matrix of partial derivatives of the following functions:

(a) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x, y)$
(b) $f\colon \mathbb{R}^2 \to \mathbb{R}^3$, $f(x, y) = (xe^y + \cos y, x, x + e^y)$
(c) $f\colon \mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (x + e^z + y, yx^2)$
(d) $f\colon \mathbb{R}^2 \to \mathbb{R}^3$, $f(x, y) = (xye^{xy}, x \sin y, 5xy^2)$

8. Compute the matrix of partial derivatives of

(a) $f(x, y) = (e^x, \sin xy)$
(b) $f(x, y, z) = (x - y, y + z)$
(c) $f(x, y) = (x + y, x - y, xy)$
(d) $f(x, y, z) = (x + z, y - 5z, x - y)$

9. Where does the plane tangent to $z = e^{x-y}$ at $(1, 1, 1)$ meet the z axis?

10. Why should the graphs of $f(x, y) = x^2 + y^2$ and $g(x, y) = -x^2 - y^2 + xy^3$ be called "tangent" at $(0, 0)$?

11. Let $f(x, y) = e^{xy}$. Show that $x(\partial f/\partial x) = y(\partial f/\partial y)$.

12. Use the linear approximation to approximate a suitable function $f(x, y)$ and thereby estimate the following:

(a) $(0.99e^{0.02})^8$
(b) $(0.99)^3 + (2.01)^3 - 6(0.99)(2.01)$
(c) $\sqrt{(4.01)^2 + (3.98)^2 + (2.02)^2}$

13. Compute the gradients of the following functions:

(a) $f(x, y, z) = x \exp(-x^2 - y^2 - z^2)$ (Note that $\exp u = e^u$.)
(b) $f(x, y, z) = \dfrac{xyz}{x^2 + y^2 + z^2}$
(c) $f(x, y, z) = z^2 e^x \cos y$

14. Compute the tangent plane at $(1, 0, 1)$ for each of the functions in Exercise 13. [The solution to part (c) only is in the Study Guide.]

15. Find the equation of the tangent plane to $z = x^2 + 2y^3$ at $(1, 1, 3)$.

16. Calculate $\nabla h(1, 1, 1)$ if $h(x, y, z) = (x + z)e^{x-y}$.

17. Let $f(x, y, z) = x^2 + y^2 - z^2$. Calculate $\nabla f(0, 0, 1)$.

18. Evaluate the gradient of $f(x, y, z) = \log(x^2 + y^2 + z^2)$ at $(1, 0, 1)$.

19. Describe all Hölder-continuous functions with $\alpha > 1$ (see Exercise 25, Section 2.2). (HINT: What is the derivative of such a function?)

20. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. What is the derivative of f?

2.4 Introduction to Paths and Curves

In this section, we introduce some of the basic geometry and computational methods for paths in the plane and space. This will be an important ingredient for the chain rule treated in the next section. We will return to paths with additional topics in Chapter 4.

Paths and Curves

One often thinks of a curve as a line drawn on paper, such as a straight line, a circle, or a sine curve. It is useful to think of a curve C mathematically as the set of values of a function that maps an interval of real numbers into the plane or space. We shall call such a map a ***path***. We usually denote a path by $\mathbf{c}$. The image C of the path then corresponds to the curve we see on paper (see Figure 2.4.1). Often we write t for the independent variable and imagine it to be *time*, so that $\mathbf{c}(t)$ is the position at time t of a moving particle, which ***traces out*** a curve as t varies. We also say $\mathbf{c}$ ***parametrizes*** C. Strictly speaking, we should distinguish between $\mathbf{c}(t)$ as a *point* in space and as a *vector* based at the origin.

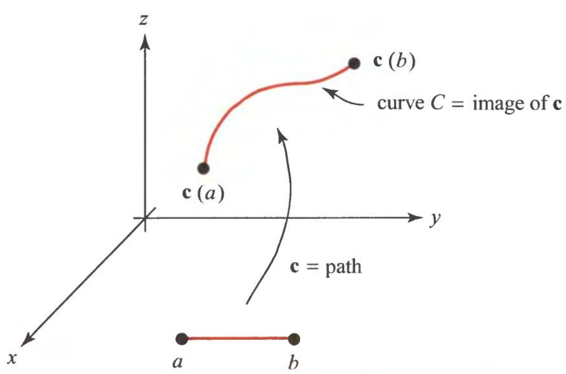


Figure 2.4.1 The map $\mathbf{c}$ is the path; its image C is the curve we "see."

EXAMPLE 1 The straight line L in $\mathbb{R}^3$ through the point (x_0, y_0, z_0) in the direction of vector $\mathbf{v}$ is the image of the path

$$\mathbf{c}(t) = (x_0, y_0, z_0) + t\mathbf{v}$$

for $t \in \mathbb{R}$ (see Figure 2.4.2). Thus, our notion of curve includes straight lines as special cases. ▲